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# Coherent and Incoherent Phase Retrieval using Gaussian Beam Modes

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**Abstract** – Phase retrieval techniques are important at terahertz frequencies where it is often difficult to determine the phase of a field directly. In this paper we present a phase retrieval technique based on a Gaussian-Beam formalism. We show how the complex mode coefficients of a coherent field can be determined by fitting to intensity distributions at any two planes. We extend the analysis to incoherent or partially-coherent fields by working in terms of coherence matrices rather than mode coefficients. Working with the coherence matrix also allows us to vary the entropy of the field as part of the fitting process and to handle the problem of noise in the measurements more correctly. We illustrate our techniques with some simple examples.

## I. INTRODUCTION

The phase retrieval problem arises when attempting to determine the phase of a complex function from measurements of intensity alone. Such situations occur frequently in fields such as microscopy and antenna design where far-field intensity patterns can most easily be measured. The complex coefficients of non-ideal horns and of antennas (e.g. planar antennas) whose mode coefficients are not otherwise known, are needed in the design of many submillimetre optical systems. Different object-plane fields can yield the same intensity in the far-field, and so to obtain a unique solution for the object-plane phase, additional information must be supplied (see e.g. Taylor[1] for an overview of the problem and possible solutions). Gerchberg and Saxton [2] suggested a technique using intensity measurements in the Fraunhofer and object planes, Misell [3] used intensity distributions in two slightly defocussed images. In the case of few-moded bolometers for example, the analysis must be applicable to partially coherent fields.

Here we follow the method of Isaak et al.[4] and describe the object in terms of Gaussian beam modes (see e.g. [5]). The mean-square error between the measured (numerically simulated) and trial intensity distributions on two planes is then minimised by adjusting the complex mode coefficients. Again using the Gaussian beam formalism, we extend the technique to partially coherent fields by adjusting the elements of its coherence matrix rather than its mode coefficients. We find it useful to describe the degree of coherence of the field, using the concept of entropy, in order to constrain the solution, particularly for the incoherent case or where noise is present in naturally coherent fields.

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## II. COHERENT PHASE RETRIEVAL

The intensity distribution of a beam, at a distance  $z$  from the waist, can be expressed in terms of basis functions ( $\psi_n$ ) appropriate to the symmetry of the source as

$$I(r, z) \propto \left| \sum_n (a_n \exp(jp_n)) \psi_n(r, z) \exp(jn\Delta\phi(z)) \right|^2, \quad (1)$$

where  $a_n$  and  $p_n$  are the mode amplitudes and phases,  $\Delta\phi$ , the phase slippage between modes is

$$\Delta\phi = \tan^{-1} \left( \frac{\lambda z}{\pi W^2} \right), \quad (2)$$

and  $W$  is the beam waist radius.

Table 1: Coherent phase retrieval

Actual Values		Retrieved Values	
$a_i$	$\Delta p_i$	$a_i$	$\Delta p_i$
0.78119	-	0.78125	-
0	(0.1000)	$1.05055 \times 10^{-4}$	-0.6233
-0.23166	0.2000	-0.23177	0.1997
0	(0.3000)	$-4.25362 \times 10^{-4}$	3.0074
0.053578	0.4000	0.053841	0.4092

So the phase retrieval problem, in the coherent case, is to determine the  $a_n$  and the  $p_n$  by fitting to intensity measurements alone. Intensity measurements on at least two planes are needed, but unlike some retrieval techniques there is no restriction on which two, since the beam can be propagated to any plane by simply changing the phase slippage term. The expression

$$\sum_k \sum_i (I_{k,i} - M_{k,i})^2 \quad (3)$$

is minimised using a suitable algorithm<sup>†</sup> to adjust the mode amplitude and phases.  $I_{k,i}$  is the modelled intensity at point  $r_i$  on plane  $k$  and  $M_{k,i}$  is the corresponding measurement. (Sampling and data-weighting issues have been discussed elsewhere [4]). As an example we have simulated the (one-dimensional) field produced by a corrugated conical horn at two planes, one close to the mouth of the horn, the other in the far-field. We sampled

<sup>†</sup> Minimisations were carried out using MATHEMATICA<sup>®</sup>'s FindMinimum routine which uses a modification of Powell's method [8].

the distributions at 11 points and attempted to recover the first 5 Hermite-polynomial modes.

The actual values used in the simulation are listed in Table 1 along with the retrieved values. The initial trial values were zero, except for the lowest-order mode amplitude which was set to 1. We found the coherent phase retrieval for examples of this type did not depend significantly on the initial values chosen. The phase retrieval process does not constrain the absolute phase and so we list the phase difference between each mode and the fundamental.

Table 1 shows excellent agreement between retrieved and simulated values. Optimisation of the choice of planes, measurement weightings and sampling may further improve the results, but in this paper we instead concentrate on extending the technique to partially coherent fields. As the method described so far relies on the field modes maintaining a fixed phase relationship, another method must be chosen for this.

An alternative approach, appropriate for both coherent and partially-coherent fields, is to work in terms of their coherence matrices rather than mode vectors. We describe such a method in the following sections.

### III. THE COHERENCE MATRIX

The analysis of partially coherent fields has been described previously [6]. We summarise some of the main points here. A partially coherent field is constructed from a set of coherent diffracting free-space modes. We assume that the field under investigation is one member of an ensemble of such fields and, as with a coherent field, it can be expanded in terms of a sum of modes

$$E^i(\mathbf{r}, \omega) = \sum_m A_m^i(\omega) \psi_m(\mathbf{r}, \omega), \quad (4)$$

so long as the bandwidth is sufficiently small that the phase at one point in a member of the ensemble is well-defined in respect to the phase at any other point in the same field. The cross spectral density  $W$  is then

$$\begin{aligned} W(\mathbf{r}', \mathbf{r}) &= \langle E^i(\mathbf{r}') E^{i*}(\mathbf{r}) \rangle \\ &= \sum_m \sum_n C_{mn} \psi_n^*(\mathbf{r}') \psi_m(\mathbf{r}) \end{aligned} \quad (5)$$

where

$$C_{mn} = \langle A_n^{i*} A_m^i \rangle \quad (6)$$

are the elements of the coherence matrix. The coherence matrix characterises the form of a field at any plane, with all the second-order statistical properties being completely specified. In terms of the cross spectral density, the elements of the coherence matrix are given by

$$C_{mn} = \int \int W(\mathbf{r}', \mathbf{r}) \psi_n^*(\mathbf{r}') \psi_m(\mathbf{r}) ds' ds \quad (7)$$

where the expansion functions  $\psi_m$  form an orthonormal basis set and  $s$  is the source.

For an incoherent field of uniform intensity

$$W(\mathbf{r}', \mathbf{r}) = I_0 \delta(\mathbf{r}' - \mathbf{r}) \quad \text{and} \quad (8)$$

$$C = I_0 I; \quad (9)$$

all the modes are excited equally and independently. For a completely coherent field the coherence matrix can be simply calculated from the ordinary mode coefficients as

$$C = AA^{*T}. \quad (10)$$

A partially coherent field can be traced through a submillimeter-wave optical system using the overall coherent-mode scattering matrix  $S$  of the system. Withington & Murphy[6] have shown that

$$D = SCS^{*T} \quad (11)$$

where  $C$  is the coherence matrix at the input plane of the system and  $D$  is the coherence matrix at the output plane. The elements of the scattering matrix for free-space propagation to a plane a distance  $z$  away are simply

$$S_{mn} = \exp(jm\Delta\phi) \delta_{mn} \quad (12)$$

where  $\Delta\phi$  is again given by (2).

Just as in the coherent case where the fields were decomposed into modes and then propagated from one plane to another, in the more general case the cross spectral density is broken down into modes and it is these modes (or elements of the coherence matrix) that are propagated. The intensity distribution of the field can then be recovered from the coherence matrix using

$$I(\mathbf{r}) = \sum_m \sum_n C_{mn} \psi_n^*(\mathbf{r}) \psi_m(\mathbf{r}). \quad (13)$$

In phase retrieval process we determine those elements of the coherence matrix that give the best fit to two measured intensity distributions. The  $N \times N$  coherence matrix of a fully coherent field (given by (10)) can be constructed from  $2N-1$  independent quantities corresponding to the  $N$  mode amplitudes and  $N-1$  phase differences of §1. Phase retrieval results using the coherence matrix formalism were similar to the mode vector method, although it was found that for some initial trial values the minimisation routine did not converge.

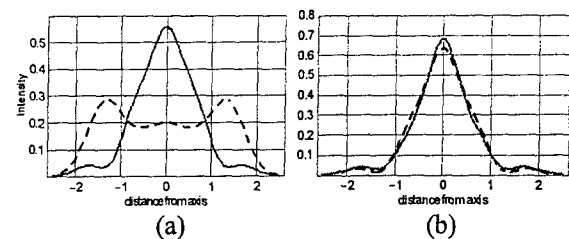


Fig. 1: Intensity (arbitrary units) as a function of off-axis distance for an incoherent beam in (a) the near- (continuous line) and far-field (dashed) and (b) in the source plane (the continuous line is the recovered intensity). 5 modes were used and the intensity was sampled at 21 points along the two planes plotted in (a).

For an incoherent or partially coherent field, however, the only restriction we have placed on the matrix is that it is

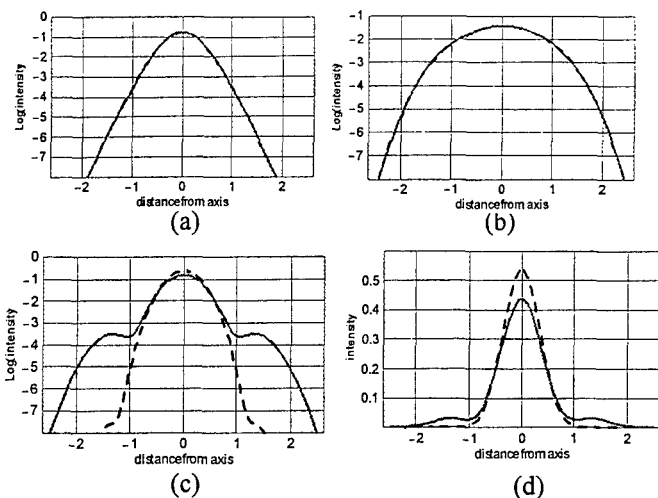


Fig. 2: Intensity (arbitrary units) as a function of off axis distance for a beam in the (a) near- and (b) far-field and in the (c)&(d) object plane. The continuous line shows the intensity derived from the recovered coherence matrix and the dashed line the actual intensity. 5 modes were used and the intensity was sampled at 21 points along two planes.

Hermitian, and the phase retrieval fitting process was carried out with respect to  $N^2$  variables.

The coherence matrix of a fully incoherent source with an intensity distribution  $I(r)$  can be calculated using (7) with  $W(r', r) = I(r)\delta(r' - r)$ . (Despite the fact that the source itself is incoherent, correlations will exist between modes if the intensity distribution is not uniform [6]). We simulated an incoherent source with the same intensity distribution as the example in §I and carried out the retrieval process for an Hermitian coherence matrix. The simulated intensities in the near- and far-field, along with the retrieved and simulated intensities in the source plane are plotted in Fig. 1. The elements of the retrieved coherence matrix differed from the actual values by a few percent.

Surprisingly, the success of this technique was found to decrease with increasing beam coherence. Fig.2 illustrates the results for a completely coherent beam (again making no restriction on the coherence matrix other than it be Hermitian). A coherence matrix that described the field well at the two sampled planes was found and yet this solution was not a good approximation to the original coherent field. Quite often, if the solution was not constrained to be a coherent field the method converged to an incoherent one. Sampling at more than two planes improved the result but since the fit to the measured data was always very good there was no way of predicting the accuracy of the reconstructed object field.

So, for completely coherent or incoherent fields we have retrieval methods that recover good approximations to the coherence matrix. For partially coherent fields, however, the minimisation must converge to the correct solution which has a beam coherence somewhere between the two extremes.

For this we need to supply information that characterises the coherence of the beam. A coherent field has a single dominant natural mode whereas for an incoherent field

power is spread evenly amongst each of the natural modes of the field. One way of describing this distribution of power amongst field modes is by the entropy of the field.

#### IV. ENTROPY

It turns out that the entropy of the field can be easily determined once the coherence matrix is known[9]. Since the  $N \times N$  coherence matrix of the field,  $C$ , is non-negative and Hermitian, it has  $N$  real eigenvalues  $\lambda_i$ . Each eigenvalue corresponds to the total power in each natural mode of the field. The probability,  $p_i$ , that a detected photon belongs to the  $i^{\text{th}}$  eigenmode is given by

$$p_i = \frac{\lambda_i}{\sum_j \lambda_j}, \quad (\sum p_i = 1) \quad (14)$$

and the entropy ( $H$ ) of the system becomes

$$H = -\sum_{i=1} p_i \ln p_i \quad (15)$$

where the sum is taken over all possible modes ( $N$ ).

A completely coherent wavefield is specified by only one non-vanishing eigenvalue, whereas a completely incoherent field (maximum amount of disorder) has all eigenvalues the same. The entropy of a field can thus vary between 0 (fully coherent) and  $\ln(N)$  (fully incoherent). As an example The entropy of the field in Fig. 2 was 1.06 (compared to a maximum possible of 1.609) which illustrates the coherence imposed on the field due to the non-uniform intensity distribution. Diagonalising the coherence matrix and finding its eigenvalues allows the entropy of the field to be determined.

Repeated diagonalisation of the coherence matrix in a minimisation routine, however, can be computationally expensive and so we use an alternative way of determining the entropy from the coherence matrix.

It can be shown that the entropy of a wavefield can also be expressed in terms of its coherence matrix as

$$H = -\text{Tr}[C' \ln(C')] \quad (16)$$

or, using the power series expansion of natural log

$$H = \text{Tr} \left[ C' \sum_{r=1}^{\infty} \frac{1}{r} (I - C')^r \right] \quad (17)$$

where  $I$  is the identity matrix and  $C'$  is the normalised coherence matrix ( $C/\text{Tr}[C]$ ).

The value of the entropy obtained using the above power series converges reasonably quickly to the value calculated by means of diagonalising the coherence matrix. To check the rate of convergence we have used both methods to calculate the entropy of a well-behaved source of variable entropy. A Gauss-Schell source[7] has a Gaussian intensity distribution (width characterised by  $\sigma_i$ ) and a Gaussian coherence length (characterised by  $\sigma_c$ ). By changing the relative scale factors ( $\sigma_i/\sigma_c$ ) it is possible to move from having a fully coherent Gaussian through to having a fully incoherent Gaussian. We calculated the entropy of the source as a function of the

degree of coherence using both the diagonalisation and power series methods.

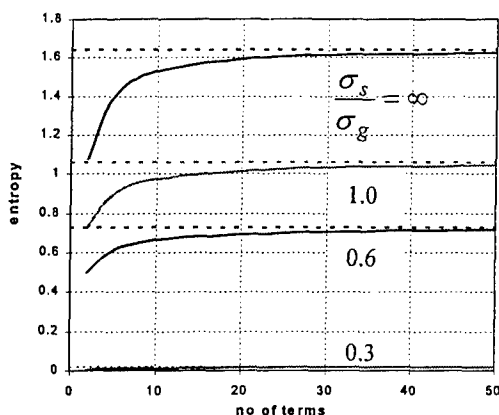


Fig. 3: Plot of entropy as a function of the number of terms in the expansion of equation (17). Entropy is calculated for four Gauss-Schell sources with different relative scale factors  $\sigma_s/\sigma_g$  ( $\infty$  for a fully incoherent source, 0 for a fully coherent source). The dashed lines show the entropy calculated by diagonalising the coherence matrix. (Since only 5 modes were used in these calculations, the maximum entropy is  $\log_e 5 = 1.609$ )

Fig. 3 shows that the power series expansion is a good way of getting a first-order measure of the degree of overall coherence efficiently.

## V. PARTIALLY COHERENT FIELDS

Finally we look at an example of a partially coherent field. We have chosen a fully coherent horn surrounded by a sheet of perfect absorber (a fully incoherent source). The coherence matrix of the whole system is simply the sum of the individual coherence matrices for the horn and absorber [6]. Fig. 4 shows the result of starting with a uniform, fully incoherent source and minimising both the intensity difference and the entropy of the field. Without the use of entropy an excellent fit to the data was found, (total error  $\sim 10^{-5}$ ) but the recovered field did not resemble the actual source field. Minimising the entropy as well as the intensity error gave a much improved estimate for the original coherence matrix.

## VI. CONCLUSION

This paper describes numerical simulations to test phase retrieval using a Gaussian-beam formalism to represent complex fields. For coherent fields fitting for mode vectors was found to be a robust technique. We have tested both coherent and partially coherent fields using a coherence matrix method. As with other techniques, success depends on the function being minimised, sampling and the initial estimate, but one significant advantage with this method is that it does allow intensity measurements to be taken across any number of planes. Here we have restricted ourselves to two. For coherent fields it was found that local minima did not vary too much from the true value, but the minimisation did not converge for some initial trial values. For partially

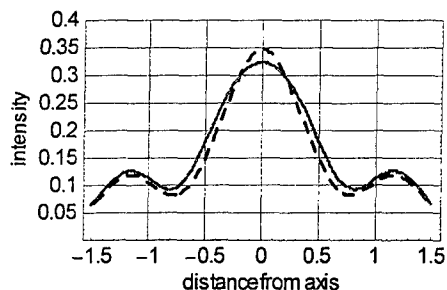


Fig. 4: Intensity (arbitrary units) as a function of off-axis distance for a coherent horn surrounded by an incoherent absorber. The continuous line shows the recovered intensity, the broken line is the actual intensity. 5 modes were used to model the field, and the intensity was sampled at 21 points along two planes.

coherent fields local minima were often found that fitted the simulated data very well but did not reproduce the object field. We found that incorporating the entropy of the beam expected into the function to be minimised improved the results.

Future work will look at the effect of measurement error on the phase retrieval technique, as well as a closer investigation of sampling, weighting of data and the best function to minimise.

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